Artificial Intelligence

III Knowledge and Reasoning

III.2 First-Order Logic

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Overview

- First-Order Logic
  - Syntax
  - Semantics
- Basic set theory
- Higher order logics

First-Order Logic

- Propositional logic assumes world contains facts
- First-order logic (FOL) assumes world contains
  - Objects as elements in the world:
    people, houses, colours, numbers, theories, wars, . . .
  - Functions mapping k objects onto one object:
    father_of, best_friend, one_more_than, head_of, . . .
  - k-ary relations between objects which are true or false:
    prime, colour, shape, brother, bigger, owns, . . .
- Variables enable use of knowledge templates
  - ➔ *Reusable* knowledge:
    instantiation with new objects gives new facts
- *Deterministic relational model*
  - Used in first-order logic reasoning, planning

FOL Syntax: Basic Elements

- **Constants**: KingJohn, 2, CardiffUniversity, ...
  (to represent objects)
- **Functions**: Sqrt, LeftLegOf, ...
  (to represent functions)
- **Predicates**: Brother, $>$, ...
  (to represent relations between objects)
- **Connectives**: $\neg$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$
  (to construct complex sentences)
- **Variables**: x, y, a, b, ...
  (to express knowledge templates)
- **Quantifiers**: $\forall$, $\exists$ (to declare variables)
- **Equality**: $=$ (to determine object equality)

FOL Syntax: Atomic Sentences

- `function`, `constant`, `predicate` are capitalised strings
- `variable` is non-capital character
- ```
  term = (function, "(", [ term, {",", term} ], ")") |
          constant | variable
  ```
  - LeftLegOf(Richard)
  - Length(LeftLegOf(x))
- ```
  atom = (predicate,"(", [ term, {",", term} ], ")") |
          term, "=", term
  ```
  - Brother(KingJohn, RichardTheLionheart)
  - $>$ (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))
  - Richard $=$ RichardTheLionheart

FOL Syntax: Complex Sentences

- Complex sentences are made from atomic sentences using connectives and quantifiers
  - ```
    sentence = (sentence,binary-connective,sentence)|
               (unary-connective,sentence)|
               (quantifier,variable,sentence)|
               atom
    ```
  - $\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$
  - $\forall x\ S(x), \quad \exists x\ S(x)$
    - $> (1,2) \vee \neg \geq (1,2)$
    - $\neg \leq (\text{Length}(\text{LeftLegOf}(\text{KingJohn})), 0)$
    - $\forall x\ \text{Sibling}(\text{KingJohn}, x) \Rightarrow \text{Sibling}(x, \text{KingJohn})$
- Parenthesis for operator precedence
    ```

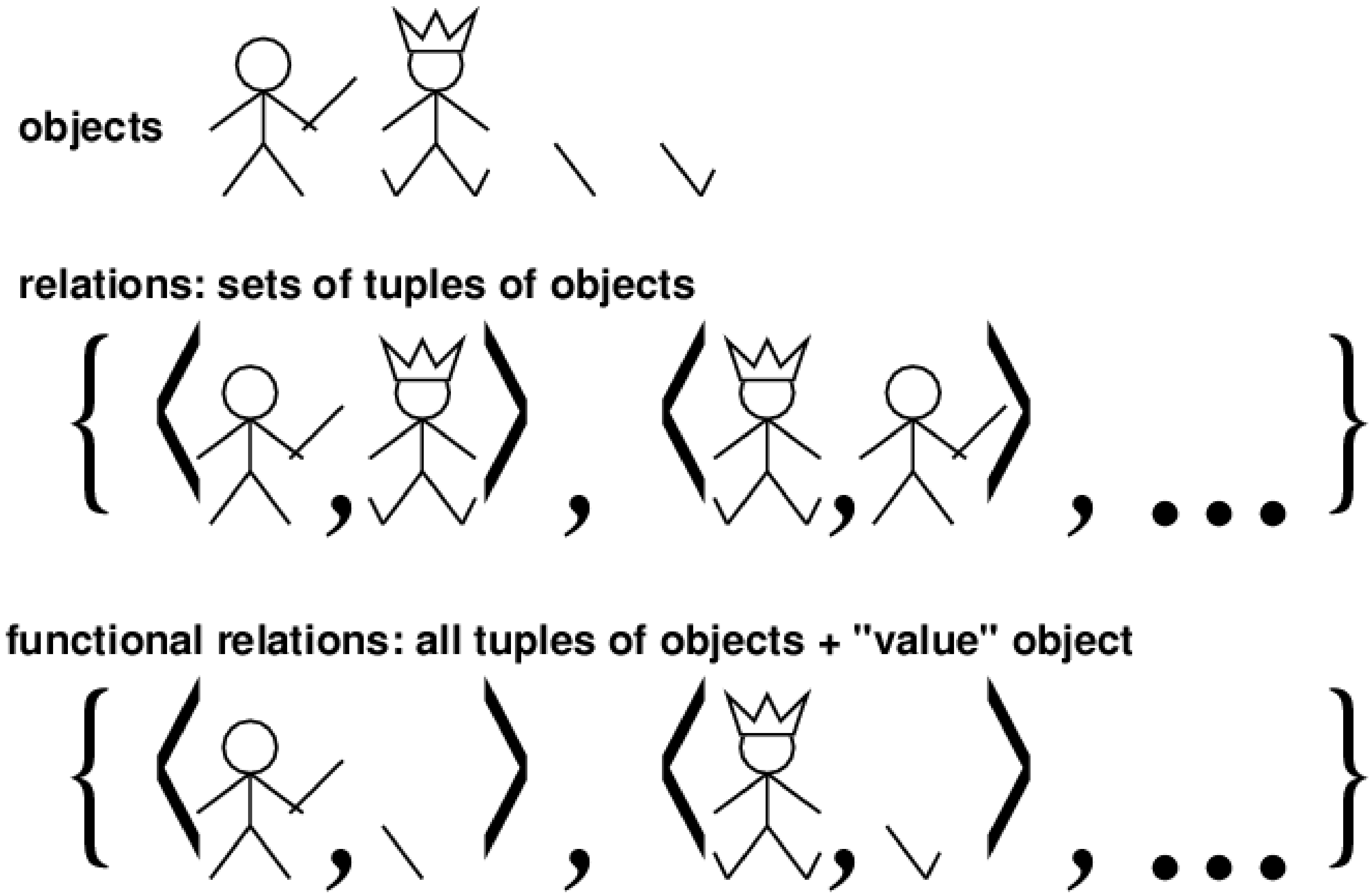
FOL Semantics: Basic Truth

- Sentences are true in a model described by
  objects & functions & relations

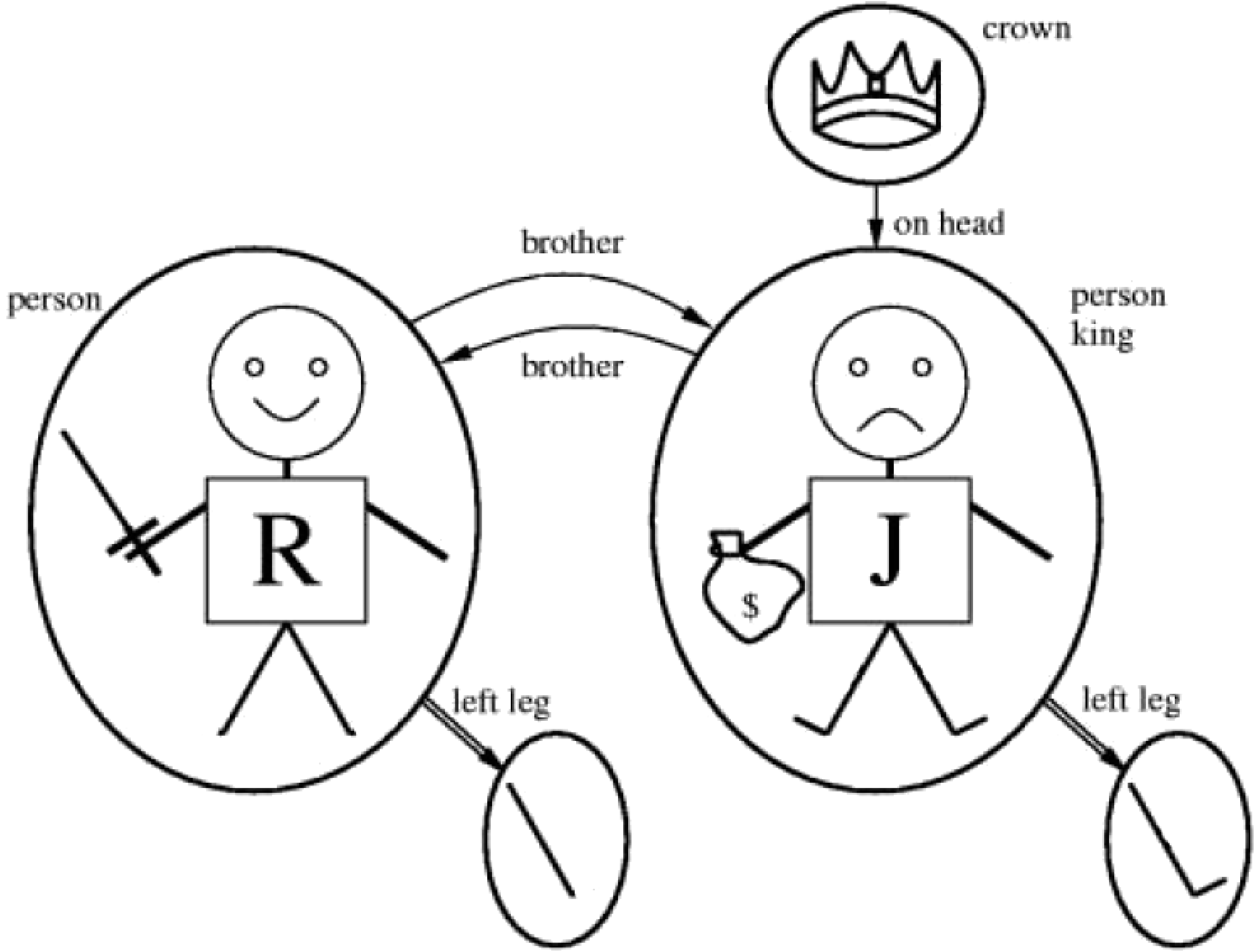
- An atomic sentence *predicate*$(term_1, \ldots, term_n)$ is *true* iff
  objects referred to by $term_1, \ldots, term_n$
  are in the relation referred to by *predicate*
  – Variables, equality and quantifiers not yet included

- *Interpretation* of model specifies referents
  - Constant symbols $\leftrightarrow$ objects in the world
  - Predicate symbols $\leftrightarrow$ relations in the world
  - Function symbols $\leftrightarrow$ functional relations in the world

FOL Model Example



FOL Model Example



Universal Quantification

- Universally quantified variables: $\forall \langle \textit{variable} \rangle \quad \langle \textit{sentence} \rangle$
  (sentence can use variable as term)
  - "Everyone at Cardiff is smart"
    $$\forall x \ \text{At}(x, \text{Cardiff}) \Rightarrow \text{Smart}(x)$$
  - $\forall x \ P$ is *conjunction of instantiations* of P
    $$\begin{aligned}&(\text{At}(\text{KingJohn}, \text{Cardiff}) \Rightarrow \text{Smart}(\text{KingJohn}))\\ \wedge\ &(\text{At}(\text{Richard}, \text{Cardiff}) \Rightarrow \text{Smart}(\text{Richard}))\\ \wedge\ &(\text{At}(\text{Cardiff}, \text{Cardiff}) \Rightarrow \text{Smart}(\text{Cardiff}))\\ \wedge\ &\ldots\end{aligned}$$
- Typically, $\Rightarrow$ is the main connective with $\forall$
  - *Common MISTAKE*: $\wedge$ is main connective with $\forall$
  - $\forall x \ \text{At}(x, \text{Cardiff}) \wedge \text{Smart}(x)$
    means "Everyone is at Cardiff and everyone is smart"!

Existential Quantification

- Existentially quantified variables: $\exists\langle\mathit{variable}\rangle \quad \langle\mathit{sentence}\rangle$
  (sentence can use variable as term)
  - "Someone at Cambridge is smart"
    $$\exists x \; \mathrm{At}(x, \mathrm{Cambridge}) \land \mathrm{Smart}(x)$$
  - $\exists x \; P$ is *disjunction of instantiations* of P
    $$\begin{aligned} &(\mathrm{At}(\mathrm{KingJohn}, \mathrm{Cambridge}) \land \mathrm{Smart}(\mathrm{KingJohn})) \\ \lor\; &(\mathrm{At}(\mathrm{Richard}, \mathrm{Cambridge}) \land \mathrm{Smart}(\mathrm{Richard})) \\ \lor\; &(\mathrm{At}(\mathrm{Cambridge}, \mathrm{Cambridge}) \land \mathrm{Smart}(\mathrm{Cambridge})) \\ \lor\; &\ldots \end{aligned}$$
- Typically, $\land$ is the main connective with $\exists$
  - *Common MISTAKE*: $\Rightarrow$ is main connective with $\exists$
  - $\exists x \; \mathrm{At}(x, \mathrm{Cambridge}) \Rightarrow \mathrm{Smart}(x)$
    is true if there is anyone who is not at Cambridge!

Properties of Quantifiers

- $\forall x\ \forall y$ is the same as $\forall y\ \forall x$
- $\exists x\ \exists y$ is the same as $\exists y\ \exists x$
  - Write $\forall x, y, \ldots$ and $\exists x, y, \ldots$ for simplicity
- $\exists x\ \forall y$ is *not* the same as $\forall y\ \exists x$
  - $\exists x\ \forall y\ \text{Loves}(x, y)$
    - "There is someone who loves everyone in the world"
  - $\forall y\ \exists x\ \text{Loves}(x, y)$
    - "Everyone in the world is loved by at least one person"
- Quantifier duality: each can be expressed using the other
  - $\forall x\ \text{Likes}(x, \text{IceCream}) \equiv \neg\exists x\ \neg\text{Likes}(x, \text{IceCream})$
  - $\exists x\ \text{Likes}(x, \text{Broccoli}) \equiv \neg\forall x\ \neg\text{Likes}(x, \text{Broccoli})$

Equality

- $term_1 = term_2$ is true under a given interpretation iff $term_1$ and $term_2$ refer to the same object
- Examples
  - $1 = 2$: true if both constants refer to the same object
  - $2 = 2$: true in any model
  - $\forall x \; *(\text{Sqrt}(x), \text{Sqrt}(x)) = x$: true in certain models
- Example: definition of (full) sibling in terms of parent
  - $\forall x, y \; \text{Sibling}(x, y) \Leftrightarrow$

$$[\quad \neg(x = y) \wedge$$
$$(\exists m, f \quad \neg(m = f) \wedge$$
$$\text{Parent}(m, x) \wedge \text{Parent}(f, x) \wedge$$
$$\text{Parent}(m, y) \wedge \text{Parent}(f, y) \quad) \quad]$$

Kinship Examples

- "Brothers are siblings"
  - $\forall x, y \; \text{Brother}(x, y) \Rightarrow \text{Sibling}(x, y)$
- "Sibling is symmetric"
  - $\forall x, y \; \text{Sibling}(x, y) \Rightarrow \text{Sibling}(y, x)$
- "One's mother is one's female parent"
  - $\forall x, y \; \text{Mother}(y) = x \Rightarrow \text{Female}(x) \wedge \text{Parent}(x, y)$
- "A first cousin is a child of a parent's sibling"
  - $\forall x, y \; \text{FirstCousin}(x, y) \Rightarrow$
    $(\exists p, s \; \text{Parent}(p, x) \wedge \text{Sibling}(s, p) \wedge \text{Parent}(s, y))$
    $\wedge \, x \neq y$

Kinship Examples

- "There is at least one child"
  - $\exists x \; \text{Child}(x)$
- "There is at most one child"
  - $\forall x, y \; \text{Child}(x) \wedge \text{Child}(y) \Rightarrow x = y$
- "There is exactly one child"
  - $\exists x \; \text{Child}(x) \wedge (\forall x, y \; \text{Child}(x) \wedge \text{Child}(y) \Rightarrow x = y)$

General Set Theory: Axioms

- General, not Zermelo-Fraenkel, sufficient for *finite* sets
- *Extensionality axiom*

$$\forall a, b, x\ (\mathrm{In}(x, a) \Leftrightarrow \mathrm{In}(x, b)) \Rightarrow a = b$$

  - Usually, $\mathrm{In}(x, a)$ is $x \in a$
- *Axiom scheme of specification* for any predicate P

$$\forall a\ \exists b\ \forall x\ \mathrm{In}(x, b) \Leftrightarrow \mathrm{In}(x, a) \wedge P(x)$$

  - Usually $b = \{x \in a : P(x)\}$
  - Special cases: $P =$ true or $P =$ false
  - *Empty set axiom*: $\exists e\ \forall x\ \neg\mathrm{In}(x, e)$
    - e is unique because of extensionality axiom
- *Adjunction axiom*

$$\forall a, b\ \exists w\ \forall x\ \mathrm{In}(x, w) \Leftrightarrow \mathrm{In}(x, a) \vee x = b$$

General Set Theory: Sets

- *List notation*: $\{x_1, x_2, \ldots, x_n\}$ denotes a set A iff

$$\forall y \; \mathrm{In}(y, A) \Leftrightarrow y = x_1 \vee y = x_2 \vee \cdots \vee y = x_n$$

  - Uniquess of A not specified
    - Extensionality guarantees uniqueness
- *Brace notation*: $\{x : P(x)\}$ denotes a set A iff

$$(\forall y \; \mathrm{In}(y, A) \Leftrightarrow P(y)) \wedge (\forall b, z \; (\mathrm{In}(z, b) \Leftrightarrow P(z)) \Rightarrow b = A)$$

  - Uniquess of A included
    - Specification guarantees that set A exists for every P
    - Extensionality guarantees second conjunct is satisfied whenever first conjuct is satisfied
  - ➔ A always exists and is unique

General Set Theory: Subsets

- *Subset*

$$\forall a, b \; \mathsf{SubsetEq}(a, b) \Leftrightarrow (\forall x \; \mathsf{In}(x, a) \Rightarrow \mathsf{In}(x, b))$$

- *Not subset*

$$\forall a, b \; \mathsf{NotSubset}(a, b) \Leftrightarrow (\exists x \; \mathsf{In}(x, a) \land \neg \mathsf{In}(x, b))$$

- *True subset*

$$\begin{aligned} \forall a, b \; \mathsf{SubsetNotEq}(a, b) \Leftrightarrow & \\ & (\forall x \; \mathsf{In}(x, a) \Rightarrow \mathsf{In}(x, b)) \land \\ & (\exists y \; \mathsf{In}(y, b) \land \neg \mathsf{In}(y, a)) \end{aligned}$$

General Set Theory: Operations

- *Union*

$$\forall a, b, x \; \text{In}(x, \text{Union}(a, b)) \Leftrightarrow \text{In}(x, a) \vee \text{In}(x, b)$$

- *Intersection*

$$\forall a, b, x \; \text{In}(x, \text{Intersection}(a, b)) \Leftrightarrow \text{In}(x, a) \wedge \text{In}(x, b)$$

- *Difference* (relative complement)

$$\forall a, b, x \; \text{In}(x, \text{Difference}(a, b)) \Leftrightarrow \text{In}(x, a) \wedge \neg \text{In}(x, b)$$

- *Power set*

$$\forall a, x \; \text{In}(x, \text{PowerSet}(a)) \Leftrightarrow \text{SubsetEq}(x, a)$$

  – Note, $|\text{PowerSet}(a)| > |a|$ (Cantor, 1891)

Russell's Paradox

- Let R be the *set of all sets that do not contain themself*

$$R = \{a : \neg \text{In}(a, a)\}$$

  - From brace notation for R (first conjunct)

$$\forall x \ \text{In}(x, R) \Leftrightarrow P(x) \qquad \text{with } P(x) = \neg \text{In}(x, x)$$

  - Hence

$$\forall x \ \text{In}(x, R) \Leftrightarrow \neg \text{In}(x, x)$$

  - Instantiate x by R

$$\text{In}(R, R) \Leftrightarrow \neg \text{In}(R, R)$$

  ➔ *Contradiction*

- Not allowing the above R is insufficient
  – Many *reciprocal* P, e.g. $P(x) = \neg \exists z \ \text{In}(x, z) \wedge \text{In}(z, x)$

  ➔ Be very vareful with things referring to themself!

Higher Order Logics

- Propositional logic: facts
- Higher-order logic
  - First-order logic: quantify over objects
  - Second-order logic: quantify over predicates
  - Higher order $k > 2$ logic:
    order k predicate has order $k - 1$ predicate arguments
    - Proofs become messy in higher-order logics
- *Type theory*: hierarchy of types assigned to entities
  - Alternative to naive set theory
    - Roughly, a type corresponds to a set
  - E.g. *typed* λ*-calculus*
    - Foundation of programming languages